

Fast optimal frictionless atom cooling in harmonic traps

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A method is proposed to cool down atoms in a harmonic trap without phase-space compression as in a perfectly slow adiabatic expansion, i.e., keeping the populations of the instantaneous initial and final levels invariant, but in a much shorter time. This may require that the harmonic trap becomes an expulsive parabolic potential in some time interval. The cooling times achieved are also shorter than previous minimal times using optimal-control bang-bang methods and real frequencies.

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A fast adiabatic expansion in a short finite time looks like a contradiction in terms. An “adiabatic” process in quantum mechanics is a slow process where the system follows at all times the instantaneous eigenvalues and eigenstates of the time-dependent Hamiltonian. This is in a sense maximally efficient as the populations do not change, i.e. there is no heating or friction, but the price to pay is that the long times needed may render the process useless or even impossible to implement. Thus, a highly desirable goal is to prepare the same final states and energies of the adiabatic process in a given finite time t_f , without necessarily following the instantaneous eigenstates along the way. We would also like the process to be robust with respect to arbitrary initial states. If fulfilled, this old goal [1] has important implications. In particular, cooling without phase-space compression, which is all that is needed for many applications other than Bose Einstein condensation, could be performed in fast cycles increasing, for example, the flux of cold atoms produced and the signal to noise ratio in an atomic clock [2], or in cold-atom pulsed beam experiments and related technology [3]. This goal also includes as a particular case a long standing question in the fields of optimal control theory and finite time thermodynamics, namely, to optimize the passage between two thermal states of a system [4, 5, 6, 7]. For time-dependent harmonic oscillators, minimal times have been established using “bang-bang” real-frequency processes believed up to now to be optimal [6], in which the frequencies are changed suddenly at certain instants but kept constant otherwise. In this letter we shall describe a robust solution to the stated general goal for atoms trapped in a time-dependent harmonic oscillator which applies both to equilibrium and non-equilibrium states. In particular we describe cooling processes performed in a time interval smaller than the minimal time of the bang-bang methods considered so far. We shall for simplicity describe our method for states representing single atoms of mass m , but the same results

are immediately applicable to N -body non-interacting fermions or to a Tonks-Girardeau gas [8], and generalizations will be relevant for other driving processes, such as cold atom launching or the transport of ultracold atoms with optical tweezers [9].

We consider an effectively one dimensional time dependent harmonic oscillator, $H = \hat{p}^2/2m + m\omega(t)^2\hat{q}^2/2$, with an initial angular frequency $\omega(0) > 0$ at time $t = 0$ and final frequency $\omega_f = \omega(t_f) < \omega(0)$ at time t_f . (This amounts to a temperature reduction by a factor $\omega_f/\omega(0)$ if the initial and final states are canonical.) The challenge is to find a trajectory $\omega(t)$ between these two values so that the populations of the oscillator levels $n = 0, 1, 2, \dots$ at t_f are equal than the ones at $t = 0$. Our main tool to engineer $\omega(t)$ and the state dynamics will be the solution of the corresponding Schrödinger equation based on the existence of invariants of motion [1, 10, 11, 12] of the form $I(t) = 1/2[(1/b^2)\hat{q}^2 m\omega_0^2 + \frac{1}{m}\hat{\pi}^2]$, where $\hat{\pi} = b(t)\hat{p} - m\dot{b}\hat{q}$ plays the role of a momentum conjugate to \hat{q}/b , the dots are derivatives with respect to time, and ω_0 is in principle an arbitrary constant. The scaling, dimensionless function $b = b(t)$ satisfies the subsidiary condition

$$\ddot{b} + \omega(t)^2 b = \omega_0^2/b^3, \quad (1)$$

an Ermakov equation where real solutions must be chosen to make I Hermitian [13]. ω_0 is frequently rescaled to unity by a scale transformation of b [1]. Other convenient choice is $\omega_0 = \omega(0)$ as we shall see. $I(t)$ has the structure of a harmonic oscillator Hamiltonian as well (as long as $\omega_0^2 > 0$), with time-dependent eigenvectors $|n(t)\rangle$ and time-independent eigenvalues $(n + 1/2)\hbar\omega_0$. The general solution of the Schrödinger equation is a superposition of orthonormal “expanding modes” $\psi(t, x) = \sum_n c_n e^{i\alpha_n(t)} \langle x|n(t)\rangle$ where $\alpha_n(t) = -(n + 1/2)\omega_0 \int_0^t dt'/b^2$, and the c_n are time independent amplitudes. For a single mode and $\omega_0^2 > 0$,

$$\Psi_n(t, x) = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4} \frac{1}{(2^n n!)^{1/2}} \exp\left[-i(n+1/2) \int_0^t dt' \frac{\omega_0}{b(t')^2}\right] \exp\left[i\frac{m}{2\hbar} \left(\frac{\dot{b}}{b(t)} + \frac{i\omega_0}{b^2}\right) x^2\right] H_n\left[\left(\frac{m\omega_0}{\hbar}\right)^{1/2} \frac{x}{b}\right],$$

with time dependent average energy

$$\langle H(t) \rangle_n = \frac{(2n+1)\hbar}{4\omega_0} \left(\dot{b}^2 + \omega(t)^2 b^2 + \frac{\omega_0^2}{b^2} \right). \quad (2)$$

The average position is zero and the standard deviation $\sigma = (\int dx x^2 |\Psi_n|^2)^{1/2}$ is proportional to b , $\sigma = b(n+1/2)^{1/2}/(m\omega_0/\hbar)^{1/2}$, which underlines the physical meaning of the scaling factor.

A much studied case corresponds to the frequency scaling $\omega(t) = \omega(0)/b^2$ with $b = (At^2 + 2Bt + C)^{1/2}$ [10, 11, 14]. Substituting this in the subsidiary condition gives $\omega_0^2 = \omega(0)^2 + AC - B^2$. For a trap with hard walls, the square-root-in-time scaling factor b ($A = 0$) has been shown to provide fast and efficient cooling [15, 16]. However, for harmonic traps, much more commonly realized in ultracold experiments, such time dependence leads to negative values of ω_0^2 even for modest cooling objectives. This makes Eq. (2) invalid and, moreover, linear combinations of a continuum of non-square-integrable expanding modes would be needed to describe the evolution of any single eigenstate of the initial trap. This is of course only a drawback to calculate the dynamics, not to realize the expansion in the laboratory. Numerical results using other (adiabatic basis) methods [17] show that, even though the root-in-time scaling is singularly efficient for adiabatic following as discussed below, the cooling performance fails for very short expansion times t_f . An alternative, successful strategy put forward here, inspired in inverse scattering techniques for complex potential optimization [18, 19, 20], is to leave $\omega(t)$ undetermined at first and impose properties on b and its derivatives at the boundaries, $t = 0$ and t_f , to assure: (a) that any eigenstate of $H(0)$ evolves as a single expanding mode and that (b) this expanding mode becomes, up to a position-independent phase factor, equal to the corresponding eigenstate of the Hamiltonian $H(t_f)$ of the final trap. This keeps the populations in the instantaneous basis equal at the initial and final times. After $b(t)$ and its derivatives are fixed at the boundaries, $b(t)$ may be chosen as a real function satisfying the boundary conditions, for example as a polynomial or some other convenient functional form with enough free parameters. Once $b(t)$ has been determined, the physical frequency $\omega(t)$ is obtained from the subsidiary equation (1).

Let us first discuss the conditions at $t = 0$. By choosing $b(0) = 1$, $\dot{b}(0) = 0$, $H(0)$ and $I(0)$ commute and have common eigenfunctions at that instant. We set $\omega_0 = \omega(0)$ from now on so that $\ddot{b}(0) = 0$ must hold as well. These boundary conditions imply that any ini-

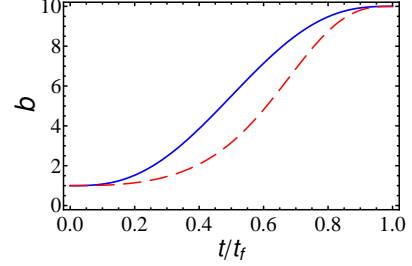


FIG. 1: (color online). Examples of ansatz for b : A simple polynomial ansatz (solid line, Eq. (3)), and an exponential of a polynomial (dashed line, $\exp \sum_{j=0}^5 d_j t^j$). $\omega(0) = 250 \times 2\pi$ Hz, $\omega(t_f) = 2.5 \times 2\pi$ Hz, $\gamma = 10$.

tial eigenstate of $H(0)$, $u_n(0)$, will evolve according to the expanding mode (2) for all later times. In general $H(t)$ and $I(t)$ will not commute for $t > 0$, so that the expanding mode $\Psi_n(t)$ may have more than one component in the “adiabatic basis” of instantaneous eigenstates of $H(t)$, $\{u_n(t)\}$, $n = 0, 1, 2, \dots$, where $u_n(t) = \left(\frac{m\omega(t)}{\pi\hbar}\right)^{1/4} \frac{1}{(2^n n!)^{1/2}} \exp\left[-\frac{m}{2\hbar}\omega(t)x^2\right] H_n\left[\sqrt{\frac{m\omega(t)}{\hbar}}x\right]$. At time t_f we want $\Psi_n(t_f)$ to be proportional, up to the global phase factor $e^{i\alpha_n(t_f)}$, to the corresponding eigenstate of the final trap $u_n(t_f)$. To this end we impose $b(t_f) = \gamma = [\omega_0/\omega_f]^{1/2}$, $\dot{b}(t_f) = 0$, $\ddot{b}(t_f) = 0$. From Eq. (2), one finds $\langle H(t_f) \rangle_n$ in terms of $b_f = b(t_f)$ and $\dot{b}_f = db(t)/dt|_{t=t_f}$. Since b_f and \dot{b}_f can be set independently we can minimize the terms depending on them separately, and the global minimum is found to be precisely at the adiabatic energy $(n+1/2)\hbar\omega_f$, which corresponds to our boundary conditions. Any other choice would necessarily produce “frictional heating”.

Substituting the simple polynomial ansatz

$$b(t) = \sum_{j=0}^5 a_j t^j \quad (3)$$

into the six boundary conditions gives six equations that can be solved to provide the coefficients, $b(t) = 6(\gamma-1)s^5 - 15(\gamma-1)s^4 + 10(\gamma-1)s^3 + 1$, where $s = t/t_f$, see Fig. 1. At initial and final times 0 and t_f , $\omega(t) = \omega_0/b(t)^2$, but this relation does not hold in general for an arbitrary intermediate time.

The above mentioned six conditions leave time dependent phases $e^{i\alpha_n(t)}$ of no relevance regarding the population of the n th level. In particular stationary density

operators with respect to $H(0)$ (e.g. a canonical state, or a pure state $|u_n(0)\rangle\langle u_n(0)|$) are mapped onto the corresponding stationary states of $H(t_f)$ with the phases canceled. In other cases the phases remain but the populations are preserved. Note that $e^{i\alpha_n(t)}$, see Eq. (2), is the phase factor that the initial state $u_n(0)$ would acquire in a virtual adiabatic process in which the adiabatic (instantaneous) energy had the form $(n + 1/2)\hbar\omega_0/b^2$. Phase control may as well be imposed by adding integral conditions such as $\tau(t_f) = \int_0^{t_f} dt \frac{1}{b(t)^2} = \frac{\omega_f t_f}{\omega_0}$, where t' is some desired time. This of course requires an ansatz more complicated than Eq. (3), such as a polynomial of higher degree.

Numerical examples of frequencies $\omega(t)$ and energies $\langle H(t) \rangle$ of fast adiabatic-like expansions are provided in Figs. 2-4 using the b shown in Fig. 1 for $\omega_0 = 250 \times 2\pi$ Hz and $\omega_f = 2.5 \times 2\pi$ Hz ($\gamma = 10$). These values can be found in actual experiments [21]. We could formally study sub-hertz frequencies ω_f but they would render the trap very sensitive to low-frequency acoustic noise [22]. Compare first the finite times considered (from 2 to 25 ms) with the times necessary for actual adiabatic following during the whole interval $0 < t < t_f$. The adiabaticity condition for the harmonic oscillator becomes $|\sqrt{2}\dot{\omega}/(8\omega^2)| \ll 1$. For a linear ramp, $\omega(t) \rightarrow \omega_0 + (\omega_f - \omega_0)t/t_f$, this implies a very long time, $t_f \gg 1.1$ s. In fact it would be necessary to expand the trap for 6 s to achieve a 1% relative error in the final energy of the ground state. A much more efficient strategy is to distribute $\dot{\omega}/\omega^2$ uniformly along the trajectory, i.e., $\dot{\omega}/\omega^2 = c$, c being constant. Thus the trap expansion speed decreases with the splitting. Solving this differential equation and imposing $\omega_f = \omega(t_f)$ we get $\omega(t) = \omega_0/[1 - (\omega_f - \omega_0)t/(t_f\omega_f)]$. This corresponds to the case $A = 0$, $2B = -(\omega_f - \omega_0)/(t_f\omega_f)$, $C = 1$ (i.e., a square-root-in-time scaling factor), and implies $t_f \gg 11$ ms from the adiabaticity condition. A 1% error level for the ground state energy is achieved after 45 ms.

A prominent feature of the trajectories, see Fig. 2b, is that $\omega(t)^2$ may be negative during some time interval in which the potential becomes an expulsive parabola. This is physically feasible and has been realized experimentally with an offset magnetic field that overcomes the optical dipole well in the axial direction of elongated cigar-shaped optical traps [23]. In general the (imaginary) frequency of the repulsive region increases for shorter cooling times as shown in Fig. 2b.

The appearance of transient energies below the final one, see e.g. the solid line in Fig. 3a near $t/t_f = 0.15$, may be misleading. It is a consequence of the repulsive regime and should not be interpreted as useful cooling in a time shorter than t_f . Since the “trap” is actually a repeller the kinetic energy would grow without bound if the potential were kept frozen at the time when the energy is minimal. Similarly, if the potential were suddenly changed into its final form, $V(t_f)$, the total energy would be higher than the adiabatic energy, i.e., the one

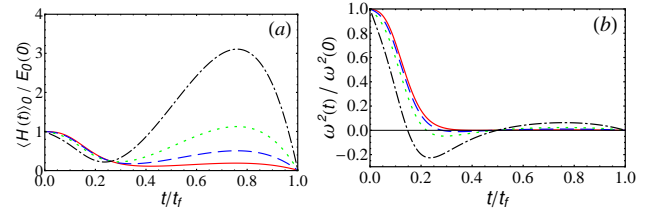


FIG. 2: (color online). (a) The average energies of the ground state expanding mode for different final times t_f : $t_f = 25$ ms (solid), $t_f = 15$ ms (dashed), $t_f = 10$ ms (dotted), and $t_f = 6$ ms (dash-dotted). Other parameters as in Fig. 1 (polynomial b) (b) The corresponding squared frequency $\omega(t)^2$.

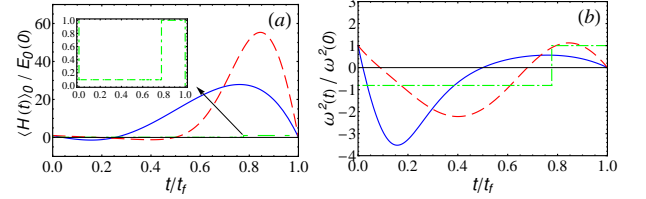


FIG. 3: (color online). Cooling in $t_f = 2$ ms. (a) Average energies of the ground expanding mode for b taken as a polynomial (solid), as an exponential of a polynomial (dashed), and for a piecewise constant frequency “bang-bang” process (dot-dashed) with $\omega_1 = i0.9\omega_0$ and $\omega_2 = \omega_0$. Other parameters as in Fig. 1. (b) The squared frequencies $\omega(t)^2$.

for a population-preserving process.

Fig. 3 illustrates that a given cooling objective may be attained in less time than the minimal time required by real-frequency bang-bang trajectories, optimal among real-frequency trajectories [6]. For the three-jump “trajectory” [6],

$$\omega(t) = \begin{cases} \omega_0 & (t = 0) \\ \omega_1 & (0 < t < \tau_1) \\ \omega_2 & (\tau_1 < t < \tau_1 + \tau_2) \\ \omega_f & (t = t_f = \tau_1 + \tau_2) \end{cases} \quad (4)$$

the smaller ω_1 and the larger ω_2 are, the faster the cooling is. Thus the fastest process to reach the target state corresponds to the limit of $\omega_1 \rightarrow 0$ and $\omega_2 \rightarrow \infty$ [6] with

$$t_f^{min} = \frac{\sqrt{1 - \omega_f/\omega_0}}{\sqrt{\omega_f\omega_0}}. \quad (5)$$

These results are based on optimal control theory, initial and final thermal states, and the constraint $\omega_{1,2} > 0$. Clearly, relaxing the positivity condition for the intermediate frequencies, makes faster processes with $t_f < t_f^{min}$ possible, which, moreover, involve only finite frequencies. Since Eq. (5) has been used to justify a finite time version of the third principle (if $\omega_f \rightarrow 0$, $t_f^{min} \rightarrow \infty$ as $\omega_f^{-1/2}$) and maximal cooling rates, the present findings call for

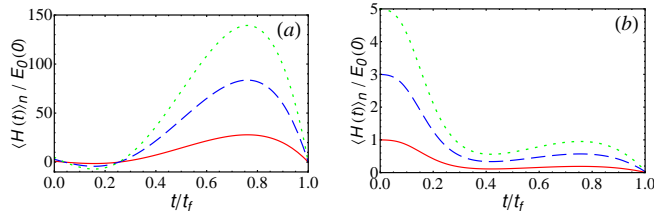


FIG. 4: (color online). Average energies for expanding modes $n = 1$ (solid), $n = 2$ (dashed), and $n = 3$ (dotted). (a) $t_f = 2$ ms; (b) $t_f = 25$ ms. Other parameters as in Fig. 1 (polynomial b).

a revision of these conclusions. A bang-bang example is shown in Fig. 3 (dot-dashed lines), for $t_f = 2$ ms, much shorter than the time $t_f^{\min} \approx 6$ ms corresponding to the initial and final frequencies chosen. $\omega_1 = i\omega_I$ is imaginary, and the corresponding $b_1(t)$ solving the Ermakov equation with initial conditions $b_1(0) = 1$ and $\dot{b}_1(0) = 1$, takes the form $b_1(t) = [1 + \frac{\omega_0^2 + \omega_I^2}{\omega_I^2} \sinh^2(\omega_I t)]^{1/2}$. In the second segment we assume ω_2 real which gives, with the final conditions $b_2(t_f) = \gamma$ and $\dot{b}_2(t_f) = 0$, $b_2(t) = \{\gamma^2 + (\frac{\omega_0^2}{\omega_2^2 \gamma^2} - \gamma^2) \sin^2[\omega_2(t - t_f)]\}^{1/2}$. The matching conditions $b_1(\tau_1) = b_2(\tau_1)$ and $\dot{b}_1(\tau_1) = \dot{b}_2(\tau_1)$ are then solved for τ_1 and t_f , see Fig. 3 and its caption for details. Of course the discontinuous jumps in this type of trajectory call into question its realizability. Fig. 3 also shows two smooth trajectories for $t_f = 2$ ms corresponding to two different ansatz for b (polynomial and exponential of a polynomial). The resulting rates of change of $\omega(t)$ are feasible with present technology. Indeed the intensity of a dipole trap (the frequency scales as the square root of the intensity) can be changed by three or four orders of magnitude in 100 ns using acousto-optics or electro-optics modulators. To monitor the sign of the square frequencies, one can superimpose two dipole beams locked respectively on the blue and red side of the line. By alternating them rapidly with a control of their relative intensity, one can shape the square frequencies and their signs at will. Alternatively, one can combine magnetic and dipole traps.

Finally, while the initial (and final) state in Figs. 2 and 3 is the ground state, Fig. 4 illustrates that the same $\omega(t)$ trajectories work as well for arbitrary excited states. This also means that fast frictionless cooling is directly applicable to arbitrary superpositions or mixed states, as well as to simple many body systems such as a polarized Fermi gas, or its (symmetrized) bosonic counterpart, the Tonks-Girardeau gas. Similar techniques may also be applicable for weakly interacting bosonic systems, the control of soliton dynamics of Bose-Einstein condensates [23, 24], and to manipulate the transport of ultracold atoms [9].

We have in summary described a method for fast, fric-

tionless cooling in a harmonic trap based on a modulation of the trap frequency $\omega(t)$ that includes in general time intervals in which the potential becomes expulsive. It is applicable to arbitrary initial states, not necessarily in equilibrium. The algorithm to design $\omega(t)$ depends on flexible ansatz functions that can be modified or made more and more complex, by adding parameters, to satisfy further requirements. This could be used, e.g., to minimize the maximal frequencies along the trajectory.

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